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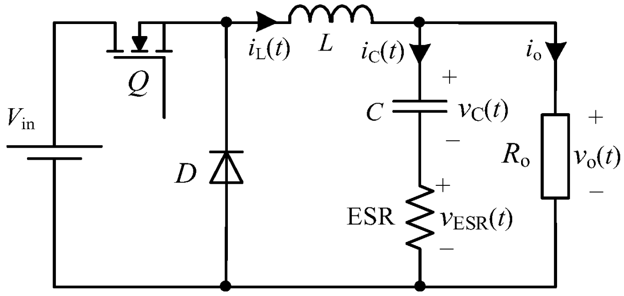
Small-signal modeling for the buck converter (assumed in CCM)

Introduction of small-signal modeling

After determining the buck converter's parameters, the small-signal modeling method is used to deduce the transfer functions of the buck converter system.

Since the buck converter behaves dynamically, it is complicated to obtain an accurate mathematical model of the converter. Scientists have proposed using small-signal modeling. The small-signal modeling first linearizes the converter at the operating point to obtain a linear time-invariant small-signal model. Then, the model is converted to the frequency domain by the Laplace Transform. Eventually, this model in the frequency domain provides transfer functions of power stage dynamics [1].

Therefore, the small-signal modeling is a linearization process that produces a linear time-invariant model for engineers to validate the stability of the switching-mode converters and design compensators conveniently.



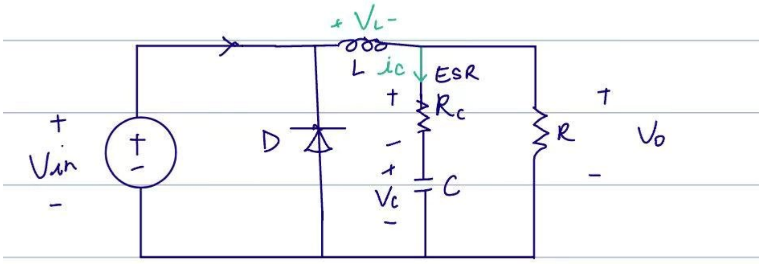
Capacitor ESR

When determining the small-signal model for a buck converter, it is necessary for engineers to consider the equivalent series resistance (ESR) of the output capacitors because the ESR provides a zero in high frequency domain. Hence, compensators should be designed to cancel out this zero.

Deduction of small-signal modeling

ON-State Equations

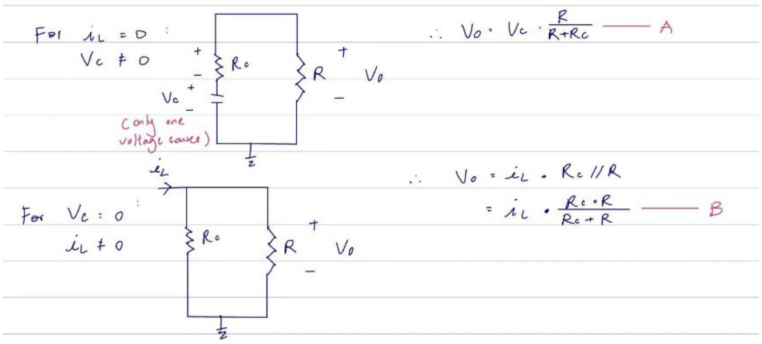
Consider when the MOSFET is switched on:



We have a voltage across the inductor:

$$V_L = V_{in} - V_o \quad (1)$$

Considering the output voltage, the superposition method is used to express V_o in terms of V_C, i_L, R, and R_C (ESR) [2]:



$$When\ i_L = 0: \quad V_o = V_c \cdot \frac{R}{(R + R_c)} \qquad (A)$$

$$When\ V_c = 0: \quad V_o = i_L \cdot \frac{(R \cdot R_c)}{(R + R_c)} \qquad (B)$$

$$\Rightarrow V_o = V_c \cdot \frac{R}{(R + R_c)} + i_L \cdot \frac{(R \cdot R_c)}{(R + R_c)} \qquad (C)$$

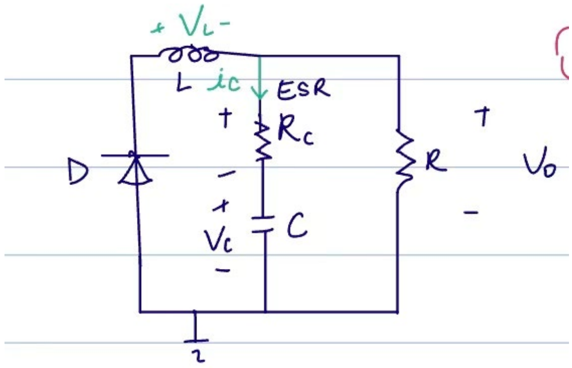
Substituting (C) into (1):

$$V_L = V_{in} - V_c \cdot \frac{R}{(R + R_c)} - i_L \cdot \frac{(R \cdot R_c)}{(R + R_c)} \qquad (D)$$

Capacitor current i_c:

$$i_c = i_L - I_o = i_L - \frac{V_o}{R} = \frac{(i_L \cdot R - V_c)}{(R + R_c)} \qquad (E)$$

OFF-State Equations



Consider when the MOSFET is switched off:

Assume that $V_D \approx 0$, as we use a low-side MOSFET instead of the diode, in which the voltage induced is very small. Then:

$$V_L = -V_o = -V_c \cdot \frac{R}{(R + R_c)} - i_L \cdot \frac{(R \cdot R_c)}{(R + R_c)} \qquad (F)$$

Similarly, the capacitor current in this stage is equivalent to (E):

$$i_c = i_L - I_o = i_L - \frac{V_o}{R} = \frac{(i_L \cdot R - V_c)}{(R + R_c)}$$

Averaging

Averaging is used to filter out signal components and noise at the switching frequency (f_s) and its harmonics. Only the DC component and the AC small signal (very low frequency and amplitude) are reserved.

$$\left(\frac{1}{T_s}\right) \int_t^{t+T_s} x(t) = \langle x(t) \rangle_{T_s}$$

Therefore, we can approximate V_L and i_c in the averaging form:

From (D): $V_L \approx \langle V_{in}(t) \rangle_{T_s} - \langle V_c(t) \rangle_{T_s} \cdot \frac{R}{(R + R_c)} - \langle i_L(t) \rangle_{T_s} \cdot \frac{(R \cdot R_c)}{(R + R_c)} \qquad (G)$

From (F): $V_L \approx -\langle V_c(t) \rangle_{T_s} \cdot \frac{R}{(R + R_c)} - \langle i_L(t) \rangle_{T_s} \cdot \frac{(R \cdot R_c)}{(R + R_c)} \qquad (H)$

From (E): $i_c \approx \frac{(\langle i_L(t) \rangle_{T_s} \cdot R - \langle V_c(t) \rangle_{T_s})}{(R + R_c)} \qquad (I)$

Let d(t) be the duty cycle.

Now we can average V_L and i_C over the period to obtain $\langle V_L(t)\rangle_{T_s} = L \cdot \frac{d\langle i_L(t)\rangle_{T_s}}{dt}$ and $\langle i_c(t)\rangle_{T_s} = C \cdot \frac{d\langle V_c(t)\rangle_{T_s}}{dt}$:

$$\langle V_L(t)\rangle_{T_s} = L \cdot \frac{d\langle i_L(t)\rangle_{T_s}}{dt} = d(t) * (\boldsymbol{G}) + \big(1 - d(t)\big) * (\boldsymbol{H})$$
$$= d(t) \cdot \langle V_{in}(t)\rangle_{T_s} - \langle V_c(t)\rangle_{T_s} \cdot \frac{R}{(R + R_c)} - \langle i_L(t)\rangle_{T_s} \cdot \frac{(R \cdot R_c)}{(R + R_c)} \qquad (J)$$

$$\langle i_c(t)\rangle_{T_s} = d(t) * (\boldsymbol{I}) + \big(1 - d(t)\big) * (\boldsymbol{I}) = \frac{\big(\langle i_L(t)\rangle_{T_s} \cdot R - \langle V_c(t)\rangle_{T_s}\big)}{(R + R_c)} \qquad (K)$$

Separating perturbation

Now we can express $\langle x(t)\rangle_{T_s}$ in terms of the DC component and AC small signals:

$$\langle x(t)\rangle_{T_s} = X + \hat{x}(t)$$

From (J):

$$L \cdot \frac{d\big(I_L + \hat{\imath}_L(t)\big)}{dt}$$
$$= \big[D + \hat{d}(t)\big] \cdot \big(V_{in} + \hat{v}_{in}(t)\big) - [V_c + \hat{v}_c(t)] \cdot \frac{R}{R + R_c} - \big(I_L + \hat{\imath}_L(t)\big) \cdot \frac{(R \cdot R_c)}{(R + R_c)}$$
$$= D \cdot V_{in} + D \cdot \hat{v}_{in}(t) + V_{in} \cdot \hat{d}(t) + \hat{d}(t) \cdot \hat{v}_{in}(t) - \left[\frac{R}{R + R_c}\right] \cdot V_c - \left[\frac{R}{R + R_c}\right] \cdot \hat{v}_c(t) - \frac{(R \cdot R_c)}{(R + R_c)} \cdot I_L - \frac{(R \cdot R_c)}{(R + R_c)} \cdot \hat{\imath}_L(t)$$

Ignoring small signal terms in the second order, such as $\hat{d}(t) \cdot \hat{v}_{in}(t)$, since its amplitude is too small. Therefore, we can categorize the above equation into the DC part and the AC (small-signal) part:

DC component: $L\frac{dI_L}{dt} = D \cdot V_{in} - \left[\frac{R}{R + R_c}\right] \cdot V_c - \frac{(R \cdot R_c)}{(R + R_c)} \cdot I_L$

AC component: $L\frac{d\hat{\imath}_L(t)}{dt} = D \cdot \hat{v}_{in}(t) + V_{in} \cdot \hat{d}(t) - \left[\frac{R}{R + R_c}\right] \cdot \hat{v}_c(t) - \frac{(R \cdot R_c)}{(R + R_c)} \cdot \hat{\imath}_L(t) \qquad (L)$

Similarly, from (K):

$$C\frac{d\big(V_c + \hat{v}_c(t)\big)}{dt} = \frac{1}{R + R_c}\cdot\ [\,R\cdot I_L + R\cdot \hat{\imath}_L(t) - V_c - \hat{v}_c(t)]$$

$$= \frac{1}{R + R_c}\cdot\ [\,R\cdot I_L - V_c] + \frac{1}{R + R_c}\cdot\ [\,R\cdot \hat{\imath}_L(t) - \hat{v}_c(t)]$$

Separate DC and AC components.

DC component: $C\frac{dV_c}{dt} = \frac{1}{R + R_c}\cdot\ (R\cdot I_L\ -\ V_c)$

AC component: $C\frac{d\hat{v}_c(t)}{dt} = \frac{1}{R + R_c}\cdot\ \big(R\cdot \hat{\imath}_L(t) - \hat{v}_c(t)\big) \qquad (M)$

Conversion small-signal model to the frequency domain

Converting equations (L) and (M) to the frequency domain (s-domain) by applying the Laplace Transform:

From (L):

$$sL\ \cdot\ \hat{\imath}_L(s) =\ D\ \cdot\ \hat{v}_{in}(s) +\ V_{in}\cdot\ \hat{d}(s) - \left[\frac{R}{(R\ +\ R_c)}\right]\cdot\ \hat{v}_c(s) - \frac{(R\ \cdot\ R_c)}{(R\ +\ R_c)}\cdot\ \hat{\imath}_L(s) \qquad (N)$$

From (M):

$$sC\ \cdot\ \hat{v}_c(s) = \frac{[R\ \cdot\ \hat{\imath}_L(s) - \hat{v}_c(s)]}{(R\ +\ R_c)} \qquad (O)$$

Making $\hat{v}_c(s)$ as the subject:

$$\hat{v}_c(s) = \frac{[R\ \cdot \hat{\imath}_L(s)]}{[(R\ +\ R_c)(sC) +\ 1]} \qquad (P)$$

It is assumed that the initial values $\hat{\imath}_L(0)$ and $\hat{v}_c(0)$ are equal to 0.

Apart from equations (N) and (O), it is needed to find the small signal of V_o to obtain the buck converter's transfer functions.

Since $V_o\ =\ V_c\cdot\frac{R}{(R + R_c)} +\ i_L\cdot\frac{(R\cdot R_c)}{(R + R_c)}$ always whatever at MOSFET ON/OFF-state,

$$\langle V_o(t)\rangle_{T_s} =\ \langle V_c(t)\rangle_{T_s}\cdot\frac{R}{(R\ +\ R_c)} +\ \langle i_L(t)\rangle_{T_s}\cdot\frac{(R\ \cdot\ R_c)}{(R\ +\ R_c)}$$

Separating perturbation:

$$V_o + \hat{v}_o(t) = [V_c + \hat{v}_c(t)] \cdot \frac{R}{R + R_c} + \frac{(R \cdot R_c)}{(R + R_c)} \cdot [I_L + \hat{\imath}_L(t)]$$

We have:

DC Part:

$$V_o = V_c \cdot \frac{R}{R + R_c} + \frac{(R \cdot R_c)}{(R + R_c)} \cdot I_L$$

AC Part:

$$\hat{v}_o(t) = \hat{v}_c(t) \cdot \frac{R}{R + R_c} + \frac{(R \cdot R_c)}{(R + R_c)} \cdot \hat{\imath}_L(t)$$

Making $\hat{v}_c(t)$ as subject:

$$\hat{v}_c(t) = \frac{\left[\hat{v}_o(t) - \frac{(R \cdot R_c)}{(R + R_c)} \cdot \hat{\imath}_L(t)\right]}{\left[\frac{R}{R + R_c}\right]}$$

Applying Laplace Transform:

$$\hat{v}_c(s) = \frac{\left[\hat{v}_o(s) - \frac{(R \cdot R_c)}{(R + R_c)} \cdot \hat{\imath}_L(s)\right]}{\left[\frac{R}{R + R_c}\right]} \qquad (Q)$$

Deduction of transfer functions

Deduction of G_{vi}(s)

By equating the equations (P) and (Q), we have:

$$\frac{[R \cdot \hat{\imath}_L(s)]}{[(R + R_c)(sC) + 1]} = \frac{\left[\hat{v}_o(s) - \frac{(R \cdot R_c)}{(R + R_c)} \cdot \hat{\imath}_L(s)\right]}{\left[\frac{R}{R + R_c}\right]}$$

Hence, we have:

$$G_{vi}(s) = \frac{\hat{v}_{o(s)}}{\hat{\imath}_{L(s)}} = \left[\frac{R}{R + R_c}\right] \cdot \left[\frac{R}{(C \cdot s \cdot (R + R_c) + 1)}\right] + \frac{(R \cdot R_c)}{(R + R_c)} \qquad (R)$$

Deduction of G_{id}(s)

Substituting equation (P) into (N):

$$sL \cdot \hat{\imath}_L(s) = D \cdot \hat{v}_{in}(s) + V_{in} \cdot \hat{d}(s) - \left[\frac{R}{(R + R_c)}\right] \cdot \frac{[R \cdot \hat{\imath}_L(s)]}{[(R + R_c)(sC) + 1]} - \frac{(R \cdot R_c)}{(R + R_c)} \cdot \hat{\imath}_L(s)$$

Setting input variables such as $\hat{v}_{in}(s)$ to 0, excepting $\hat{d}(s)$:

$$sL \cdot \hat{\imath}_L(s) = D \cdot 0 + V_{in} \cdot \hat{d}(s) - \left[\frac{R}{(R + R_c)}\right] \cdot \frac{[R \cdot \hat{\imath}_L(s)]}{[(R + R_c)(sC) + 1]} - \frac{(R \cdot R_c)}{(R + R_c)} \cdot \hat{\imath}_L(s)$$

Bring all $\hat{\imath}_L(s)$ terms to the left-hand side:

$$\hat{I}_{L(s)} \cdot \left\{ s \cdot L + \left[\frac{R^2}{((R + R_c) \cdot [s \cdot C \cdot (R + R_c) + 1])}\right] + \left(\frac{(R \cdot R_c)}{(R + R_c)}\right)\right\} = V_{in} \cdot \hat{d}(s)$$

Therefore:

$$G_{id}(s) = \frac{\hat{\imath}_L(s)}{\hat{d}(s)} = \frac{V_{in}}{\left\{ s \cdot L + \left[\frac{R^2}{((R + R_c) \cdot [s \cdot C \cdot (R + R_c) + 1])}\right] + \frac{(R \cdot R_c)}{(R + R_c)}\right\}} \qquad (S)$$

Deduction of $G_{vd}(s)$

From equation (N):

$$sL \cdot \hat{\imath}_L(s) = D \cdot \hat{v}_{in}(s) + V_{in} \cdot \hat{d}(s) - \left[\frac{R}{(R + R_c)}\right] \cdot \hat{v}_c(s) - \frac{(R \cdot R_c)}{(R + R_c)} \cdot \hat{\imath}_L(s)$$

$$\hat{\imath}_L(s) = \frac{\left[D \cdot \hat{v}_{in}(s) + V_{in} \cdot \hat{d}(s) - \left[\frac{R}{(R + R_c)}\right] \cdot \hat{v}_c(s)\right]}{sL + \frac{(R \cdot R_c)}{(R + R_c)}} \qquad (T)$$

From equation (O):

$$sC \cdot \hat{v}_c(s) = \frac{[R \cdot \hat{\imath}_L(s) - \hat{v}_c(s)]}{(R + R_c)}$$

$$\hat{\imath}_L(s) = \frac{\left[sC \cdot \hat{v}_c(s) \cdot (R + R_c) + \hat{V}_c(s)\right]}{R} \qquad (U)$$

By equating the equations (T) and (U), we have:

$$\frac{\left[D \cdot \hat{v}_{in}(s) + V_{in} \cdot \hat{d}(s) - \left[\frac{R}{(R + R_c)}\right] \cdot \hat{v}_c(s)\right]}{sL + \frac{(R \cdot R_c)}{(R + R_c)}} = \frac{[sC \cdot \hat{v}_c(s) \cdot (R + R_c) + \hat{v}_c(s)]}{R}$$

Setting input variables such as $\hat{v}_{in}(s)$ to 0, excepting $\hat{d}(s)$:

$$\frac{\left[D \cdot 0 + V_{in} \cdot \hat{d}(s) - \left[\frac{R}{(R + R_c)}\right] \cdot \hat{v}_c(s)\right]}{sL + \frac{(R \cdot R_c)}{(R + R_c)}} = \frac{[sC \cdot \hat{v}_c(s) \cdot (R + R_c) + \hat{v}_c(s)]}{R}$$

$$R \cdot \left[V_{in} \cdot \hat{d}(s) - \left[\frac{R}{(R + R_c)}\right] \cdot \hat{v}_c(s)\right] = \left[sL + \frac{(R \cdot R_c)}{(R + R_c)}\right] \cdot [sC \cdot \hat{v}_c(s) \cdot (R + R_c) + \hat{v}_c(s)]$$

$$R \cdot V_{in} \cdot \hat{d}(s) = \hat{v}_c(s) \cdot \left\{\left[sL + \frac{(R \cdot R_c)}{(R + R_c)}\right][sC \cdot (R + R_c) + 1] + \left[\frac{R^2}{(R + R_c)}\right]\right\}$$

$$\frac{\hat{v}_c(s)}{\hat{d}(s)} = \frac{R \cdot V_{in}}{\left\{\left[sL + \frac{(R \cdot R_c)}{(R + R_c)}\right][sC \cdot (R + R_c) + 1] + \left[\frac{R^2}{(R + R_c)}\right]\right\}} \qquad (V)$$

Applying the physical equation of the capacitor voltage and current relationship:

$$i_c(s) = sC \cdot \hat{v}_c(s)$$

Applying the physical equation of the capacitor voltage and output voltage relationship:

$$\hat{v}_o(s) = \hat{v}_c(s) + i_c(s) \cdot R_c$$

$$\hat{v}_o(s) = \hat{v}_c(s)(1 + sC \cdot R_c) \qquad (W)$$

Substituting (W) into (V):

$$\frac{\hat{v}_o(s)}{\hat{d}(s)} = \frac{R \cdot V_{in} \cdot (1 + sC \cdot R_c)}{\left\{\left[sL + \frac{(R \cdot R_c)}{(R + R_c)}\right][sC \cdot (R + R_c) + 1] + \left[\frac{R^2}{(R + R_c)}\right]\right\}}$$

Validating transfer functions

To validate the deducted transfer functions, we may compare them with the transfer functions provided in online [3]:

(2) CCM 小信号传递函数

$$\hat{v}_o(s) = G_{vd}(s)\times\hat{d}(s) + G_{vg}(s)\times\hat{v}_g(s) - Z_{out}(s)\times\hat{i}_o(s)$$

$$\hat{i}_L(s) = G_{id}(s)\times\hat{d}(s) + G_{ig}(s)\times\hat{v}_g(s) + G_{ii}(s)\times\hat{i}_o(s)$$

$$G_{vd}(s) = V_g\frac{1+s/w_{zc}}{1+s/Qw_0+s^2/w_o^2} = G_{vd0}\frac{1+s/w_{zc}}{1+s/Qw_0+s^2/w_o^2}$$

$$G_{vg}(s) = D\frac{1+s/w_{zc}}{1+s/Qw_0+s^2/w_o^2} = G_{vg0}\frac{1+s/w_{zc}}{1+s/Qw_0+s^2/w_o^2}$$

$$Z_{out}(s) = R_L\frac{(1+s/w_{zL})(1+s/w_{zc})}{1+s/Qw_0+s^2/w_o^2}$$

$$G_{id}(s) = \frac{V_g}{R}\frac{1+s/w_{zp}}{1+s/Qw_0+s^2/w_o^2} = G_{id0}\frac{1+s/w_{zp}}{1+s/Qw_0+s^2/w_o^2}$$

$$G_{ig}(s) = \frac{D}{R}\frac{1+s/w_{zp}}{1+s/Qw_0+s^2/w_o^2} = G_{ig0}\frac{1+s/w_{zp}}{1+s/Qw_0+s^2/w_o^2}$$

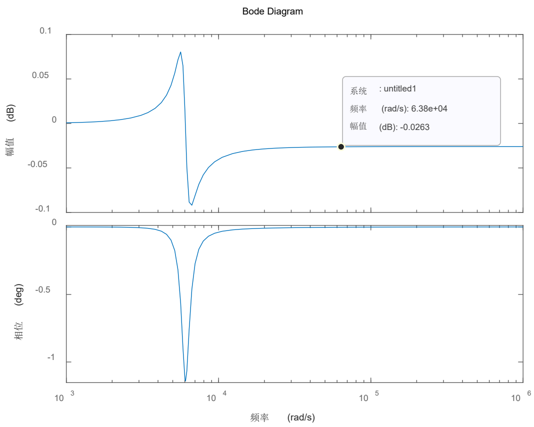
$$G_{ii}(s) = \frac{1+s/w_{zc}}{1+s/Qw_0+s^2/w_o^2}$$

其中: $w_0 = \frac{1}{\sqrt{LC}}$, $Q = \frac{1}{w_0[L/R+(R_L+R_C)C]}$

$w_{zL} = \frac{R_L}{L}$, $w_{zc} = \frac{1}{R_C C}$, $w_{zp} = \frac{1}{RC}$

Dividing the deduced $G_{vd}(s)$ by the $G_{vd\_SS}(s)$ in a Bode plot:

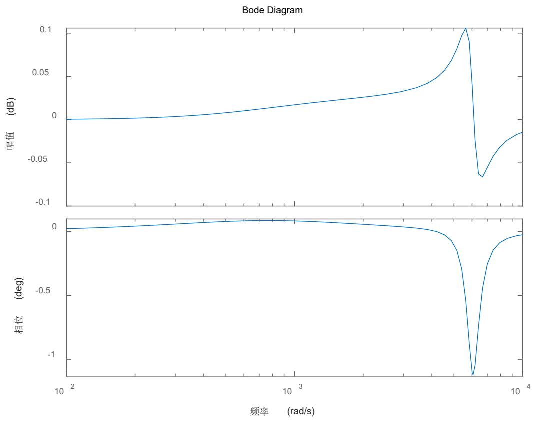
```
s = tf('s');
Ts = 1/200e3;

%Cisco's calculation
Vin = 24;
L = 210e-6;
R = 10;
Rc = 30e-3;
C = 130e-6;
wzc = 1/(Rc*C);
wzp = 1/(R*C);
wo = 1/sqrt(L*C);
Q = 1/(wo*(L/R+(Rc)*C));


% Gvd deduced from small-signal modelling by hands.
Gvd = (R*Vin*(1+s*C*Rc))/((s*L+(R*Rc)/(R+Rc))*(s*C*(R+Rc)+1)+(R*R/(R+Rc)));

%This is Gvd provided from Internet
Gvd_SS = Vin*(1+s/wzc)/(1+s/(Q*wo)+s*s/(wo*wo));
% bode(Gvd_PLECS,Gvd_SS)

bode(Gvd/Gvd_SS)
```



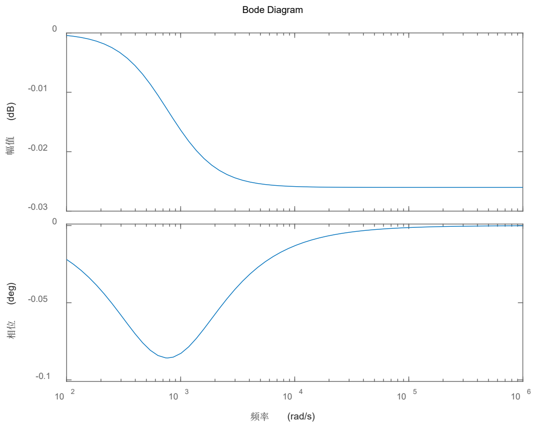
It is observed that the deduced transfer function $G_{vd}(s)$ is approximately equivalent to that provided online since the result has a gain of 0 dB (gain of 1) and a phase shift of 0°, indicating that the deduced $G_{vd}(s)$ holds true.

Similarly, the deduced $G_{vi}(s)$ and $G_{vi}(s)$ hold true:

```
s = tf('s');
Ts = 1/200e3;

%Cisco's calculation
Vin = 24;
L = 210e-6;
R = 10;
Rc = 30e-3;
C = 130e-6;
wzc = 1/(Rc*C);
wzp = 1/(R*C);
wo = 1/sqrt(L*C);
Q = 1/(wo*(L/R+(Rc)*C));

% Gid deduced from small-signal modelling by hands.
Gid = Vin/(L*s+R*R/((R+Rc)*(1+C*s*(R+Rc)))+R*Rc/(R+Rc));

%This is Gid provided from Internet
Gid_SS = Vin/R * (1+s/wzp)/(1+s/(Q*wo)+s*s/(wo*wo));

%This is Gid provided from Internet (no ESR)
Gid_no_esr = Vin*(C*s+1/R)/(L*C*s*s+L*s/R+1);

bode(Gid/Gid_SS)
```




```
s = tf('s');
Ts = 1/200e3;

%Cisco's calculation
Vin = 120;
L = 210e-6;
R = 10;
Rc = 30e-3;
C = 130e-6;
wzc = 1/(Rc*C);
wzp = 1/(R*C);
wo = 1/sqrt(L*C);
Q = 1/(wo*(L/R+(Rc)*C));


% Gvi deduced from small-signal modelling by hands.
Gvi = R*R / ((R+Rc)*((1+C*s*(R+Rc)))) + R*Rc/(R+Rc);

%This is Gvi provided from Internet
Gvi_SS = R * (1+s/wzc)/(1+s/wzp);

%This is Gvi provided from Internet (no ESR)
Gvi_no_esr = 1/(C*s+1/R);

bode(Gvi/Gvi_SS)
```



Therefore, it is reliable to use the deduced transfer functions to design compensators according to different control methods.